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Strong uniqueness of the restricted Chebyshev center with respect to an *RS*-set in a Banach space☆

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Let G be a strict *RS*-set (*resp*. an *RS*-set) in X and let F be a bounded (*resp.* totally bounded) subset of X satisfying $r_G(F) > r_X(F)$, where $r_G(F)$ is the restricted Chebyshev radius of F with respect to G. It is shown that the restricted Chebyshev center of F with respect to G is strongly unique in the case when X is a real Banach space, and that, under some additional convexity assumptions, the restricted Chebyshev center of F with respect to G is strongly unique of order $\alpha \geqslant 2$ in the case when X is a complex Banach space.

Keywords: RS-set; Restricted Chebyshev center; Strong uniqueness

1. Introduction

Let X be a Banach space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$, the reals, or $\mathbb{F} = \mathbb{C}$, the complex plane. Let G be a closed nonempty subset of X. For a bounded subset F of X, an element $g_0 \in G$ is called a restricted Chebyshev center of F with respect to G (or a best simultaneous approximation to F from G) if it satisfies that

$$\sup_{x\in F} \|x - g_0\| \leqslant \sup_{x\in F} \|x - g\| \quad \text{for each } g \in G.$$

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The set of all restricted Chebyshev centers of F with respect to G is denoted by $P_G(F)$, that is,

$$P_G(F) = \left\{ g_0 \in G : \sup_{x \in F} \|x - g_0\| = r_G(F) \right\},$$

where $r_G(F)$ is the restricted Chebyshev radius of F with respect to G defined by

$$r_G(F) = \inf_{g \in G} \sup_{x \in F} \|x - g\|.$$

Motivated by the work of Rozema and Smith [16], Amir [1] introduced the concept of an *RS*-set in a real Banach space and then gave the uniqueness results for the restricted Chebyshev center with respect to an *RS*-set. Recently, there are several papers concerned with the uniqueness of the best approximation from an *RS*-set, see, e.g., [9,13,14]. Surprisingly, the strong uniqueness of the restricted Chebyshev center with respect to an *RS*-set have not been paid attention so far although it has been studied for the case when G is an interpolating subspace in [8,10] independently. The first part of the present paper is to generalize the strong uniqueness results on the case when G is an interpolating subspace to the case when G is an *RS*-set. As will be seen in Section 3, this generalization is not trivial. In fact, for this end, we need to establish a general strong uniqueness theorem for the restricted Chebyshev center with respect to a polyhedron of finite dimension in a Banach space.

On the other hand, motivated by the work in real Banach spaces, one problem may be of interest: can one develop a similar theory for *RS*-sets in complex Banach spaces? This problem has never been considered before. It is not difficult to define an *RS*-set in a complex Banach space similar to one in a real space. However, when we try to establish the same strong uniqueness results for an *RS*-set in a complex Banach space, we find that it is completely different from that in a real space. First, the restricted Chebyshev center with restrict to an *RS*-set in a complex Banach space may not be unique in general; and secondly, the restricted Chebyshev center is not strongly unique even in the case when it is unique. The second part of the present paper is to establish some results on strong uniqueness of order $\alpha \geqslant 2$ for the restricted Chebyshev center with respect to an *RS*-set in a complex Banach space.

We conclude the section by describing the organization of this paper. In the next section, we use the notion of the strong CHIP, which is taken from optimization theory and plays an important role there, to verify the characterization theorem of the restricted Chebyshev center with respect to an *RS*-set and some basic facts, which are used in other sections. In Section 3, we consider the strong uniqueness of the restricted Chebyshev center with respect to an *RS*-set in a real Banach space. We first establish a general strong uniqueness theorem for the restricted Chebyshev center with respect to a polyhedron of finite dimension and then prove that, for any totally bounded subset (*resp.* bounded subset) F of X with $r_G(F) > r_X(F)$, the restricted Chebyshev centers of F with respect to G is strongly unique provided that G is an *RS*-set (*resp.* a strict *RS*-set) in a real Banach space. Finally, in the last section, the uniqueness and the generalized strong uniqueness of the restricted Chebyshev center with respect to an *RS*-set in a complex Banach space are studied. We first give an counter-example to illustrate that, for a totally bounded subset F satisfying $r_G(F) > r_X(F)$, the restricted Chebyshev center with respect to an *RS*-set is not unique in a complex Banach space. Then, under some additional convexity assumptions, we show that, for any totally bounded subset (*resp.* bounded subset) F of X with $r_G(F) > r_X(F)$, the restricted Chebyshev centers of F with respect to G is strongly unique of order $\alpha \geqslant 2$ if G is an *RS*-set (*resp.* a strict *RS*-set) in a complex Banach space.

2. Preliminaries and characterizations

We begin with some basic notations, most of which is standard (cf. [3]). In particular, for a set A in a Banach space, the interior (*resp.* closure, convex hull, convex cone hull, linear hull, boundary) of A is defined by int A (*resp.* $\overline{A}$, conv A, cone A, span A, bd A). Also we adopt the convention that $\mathrm{Re}\,\alpha = \alpha$ in the case when α is a real number.

Let Y be a subspace of X. For a nonempty convex closed subset C of Y, the normal cone $N_C(x)$ of C at x is defined by

$$N_C(x) = \{z^* \in Y^* : \mathrm{Re}\langle z^*, y - x\rangle \leqslant 0 \quad \text{for all } y \in C\}. \tag{2.1}$$

Let $x \in \bigcap_{i=1}^m C_i$. Thus, following [5,11], a collection $\{C_1, C_2, \ldots, C_m\}$ of convex closed sets in Y is called to have the strong conical hull intersection property (CHIP) at x if and only if

$$N_{\bigcap_{i=1}^m C_i}(x) = \sum_{i=1}^m N_{C_i}(x). \tag{2.2}$$

Let f be a proper convex continuous function defined on Y. Then, as in [11], the subdifferentiable of f at x is denoted by $\partial f(x)$ and defined by

$$\partial f(x) := \{z^* \in Y^* : \quad f(x) + \mathrm{Re}\langle z^*, y - x\rangle \leqslant f(y) \quad \text{for all } y \in Y\}. \tag{2.3}$$

Let $\mathbf{B}^*$ denote the closed unit ball of the dual X^* and ext $\mathbf{B}^*$ the set of all extreme points from $\mathbf{B}^*$. Let $\mathbf{B}^*$ be endowed with the weak*-topology. Then $\mathbf{B}^*$ is a compact Hausdorff space. We use $\overline{\mathrm{ext}\,\mathbf{B}^*}$ to denote the weak*-closure of the set ext $\mathbf{B}^*$. For a bounded subset F of X, define

$$U_F(x^*) = \sup_{x \in F} \mathrm{Re}\langle x, x^*\rangle \quad \text{for each } x^* \in \mathbf{B}^* \tag{2.4}$$

and

$$U_F^+(x^*) = \inf_{O \in N(x^*)} \sup_{u^* \in O} U_F(u^*) \quad \text{for each } x^* \in \mathbf{B}^*, \tag{2.5}$$

where $N(x^*)$ denotes the set of all open neighborhoods around x^* in $\mathbf{B}^*$. From [7], see also [19,20], we have the following proposition:

Proposition 1. *Let F be a bounded subset of X and let U_F^+ be defined by* (2.5). *Then*

(i) *U_F^+ is upper semi-continuous on $\mathbf{B}^*$ and*

$$\sup_{x^* \in \mathbf{B}^*} \left\{U_F^+(x^*) - \mathrm{Re}\langle x^*, g\rangle\right\} = \sup_{x \in F} \|x - g\| \quad \textit{for each } g \in X; \tag{2.6}$$

(ii) *if F is totally bounded, U_F is continuous and $U_F^+ = U_F$ on $\mathbf{B}^*$.*

Lemma 1. *Let $g \in X$. Then*

$$\sup_{x^* \in \overline{\text{ext}\,\mathbf{B}^*}} \left\{ U_F^+(x^*) - \text{Re}\langle x^*, g\rangle \right\} = \sup_{x \in F} \|x - g\|. \tag{2.7}$$

Proof. By the well-known Krein–Milman theorem, we have that

$$\begin{aligned} \sup_{x \in F} \|x - g\| &= \sup_{x^* \in \mathbf{B}^*} \left\{ U_F(x^*) - \text{Re}\langle x^*, g\rangle \right\} \\ &= \sup_{x^* \in \overline{\text{ext}\,\mathbf{B}^*}} \left\{ U_F(x^*) - \text{Re}\langle x^*, g\rangle \right\} \\ &\leqslant \sup_{x^* \in \overline{\text{ext}\,\mathbf{B}^*}} \left\{ U_F^+(x^*) - \text{Re}\langle x^*, g\rangle \right\} \\ &\leqslant \sup_{x^* \in \mathbf{B}^*} \left\{ U_F^+(x^*) - \text{Re}\langle x^*, g\rangle \right\} \\ &= \sup_{x \in F} \|x - g\|. \end{aligned}$$

This completes the proof. □

Let $g \in X$ and let F be a bounded subset of X. Define

$$M_{F-g} = \left\{ x^* \in \overline{\text{ext}\,\mathbf{B}^*} : \ U_F^+(x^*) - \text{Re}\langle x^*, g\rangle = \sup_{x \in F} \|x - g\| \right\}; \tag{2.8}$$

$$E_{F-g} = \left\{ x^* \in \text{ext}\,\mathbf{B}^* : \ U_F(x^*) - \text{Re}\langle x^*, g\rangle = \sup_{x \in F} \|x - g\| \right\}. \tag{2.9}$$

Note that, by Lemma 2.1, M_{F-g} is a nonempty weak* compact set. Furthermore, if F is totally bounded, so is E_{F-g}.

Let f_F denote the convex function on Y defined by

$$f_F(g) = \sup_{x^* \in \overline{\text{ext}\,\mathbf{B}^*}} \left\{ U_F^+(x^*) - \text{Re}\langle x^*, g\rangle \right\}, \quad g \in Y. \tag{2.10}$$

For a subset M of X^*, let

$$M|_Y = \{ z^*|_Y \in Y^* : z^* \in M \}, \tag{2.11}$$

where $z^*|_Y$ denotes the restriction of the functional z^* on Y. Then the subdifferential of the function f_F is given by the following lemma:

Lemma 2. *Suppose that Y is a finite-dimensional subspace of X. Then*

$$\partial f_F(g) = -\text{co}\, M_{F-g}|_Y. \tag{2.12}$$

In addition, if F is totally bounded,

$$\partial f_F(g) = -\text{co}\, E_{F-g}|_Y. \tag{2.13}$$

Proof. Let $x^* \in \overline{\text{ext}\,\mathbf{B}^*}$ and define

$$f_{F,x^*}(g) = U_F^+(x^*) - \text{Re}\langle x^*, g\rangle \quad \text{for each} \quad g \in Y. \tag{2.14}$$

By the definition, it is easy to show that the subdifferential of f_{F,x^*} at $g \in Y$

$$\partial f_{F,x^*}(g) = -x^*|_Y. \tag{2.15}$$

It follows from [12, Theorem 3.1] that

$$\partial f_F(g) = \text{co}\left\{\partial f_{F,x^*}(g) : \ f_{F,x^*}(g) = \sup_{z^* \in \overline{\text{ext}\,\mathbf{B}^*}} f_{F,z^*}(g)\right\}. \tag{2.16}$$

Clearly, $M_{F-g} = \{x^* \in \overline{\text{ext}\,\mathbf{B}^*} : \ f_{F,x^*}(g) = \sup_{z^* \in \overline{\text{ext}\,\mathbf{B}^*}} f_{F,z^*}(g)\}$. Hence (2.12) follows from (2.15) and (2.16).

Now assume that F is totally bounded. Then, by Proposition 2.1(ii), for each $x^* \in M_{F-g}$, we have that

$$U_F(x^*) - \text{Re}\langle x^*, g\rangle = \sup_{x \in F} \|x - g\|. \tag{2.17}$$

Take $\bar{x} \in \overline{F}$ such that $\text{Re}\langle x^*, \bar{x}\rangle - \text{Re}\langle x^*, g\rangle = \sup_{x\in F} \|x - g\|$. By Singer [17, Lemma 1.3, p. 169], there exist $x_1^*, \ldots, x_k^* \in \text{ext}\,\mathbf{B}^*$ $(1 \leqslant k \leqslant 2 \dim Y)$ and positive numbers $\lambda_1, \ldots, \lambda_k$ with $\sum_{i=1}^k \lambda_i = 1$ such that

$$\langle x^*, \bar{x}\rangle = \sum_{i=1}^k \lambda_i \langle x_i^*, \bar{x}\rangle \quad \text{and} \quad x^*|_Y = \sum_{i=1}^k \lambda_i x_i^*|_Y. \tag{2.18}$$

This with (2.17) implies that $x_i^* \in E_{F-g}$ for any $i = 1, 2, \ldots, k$; hence $x^*|_Y \in \text{co}\, E_{F-g}|_Y$ and (2.13) is proved. □

Let $\{y_1, y_2, \ldots, y_n\}$ be n linearly independent elements of X. Define

$$G = \left\{g = \sum_{i=1}^n c_i y_i : \quad c_i \in J_i\right\}, \tag{2.19}$$

where each J_i is a subset of the field $\mathbb{F}$ of one of the following types:

(I) the whole of $\mathbb{F}$;
(II) a nontrivial proper convex closed (bounded or unbounded) subset of $\mathbb{F}$ with nonempty interior;
(III) a singleton of $\mathbb{F}$.

Let Z denote the subspace spanned by $\{y_1, y_2, \ldots, y_n\}$. For each i, define the linear functional c_i on Z by

$$c_i(g) = c_i \quad \text{for each} \quad g = \sum_{i=1}^n c_i y_i. \tag{2.20}$$

Let $I = \{1, 2, \ldots, n\}$ and let I_0 and I_1 denote the index sets of all i such that J_i is of the type (III) and (II), respectively. Set

$$I(g) = \{i \in I_1 : \; c_i(g) \in \text{bd } J_i\},$$

$$\sigma_i(g) = -N_{J_i}(c_i(g)) \setminus \{0\} \quad \text{for each } i \in I_1$$

and

$$Y = \{g \in Z : \;\; c_i(g) = 0 \; \forall i \in I_0\}.$$

The following theorem gives the characterization of a restricted Chebyshev center with respect to the set G given by (2.19).

Theorem 1. *Suppose that G is defined by* (2.19) *and $F \subset X$ is a bounded subset. Let $g_0 \in G$. Then $g_0 \in P_G(F)$ if and only if there exist $A(F - g_0) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subseteq M_{F-g_0}$, $B(g_0) = \{i_1, i_2, \ldots, i_m\} \subseteq I(g_0), \sigma_{i_j} \in \sigma_{i_j}(g_0)$, $j = 1, \ldots, m$ (with $1 + m \leqslant k + m \leqslant \dim Y + 1$ if $\mathbb{F} = \mathbb{R}$ and $1 + m \leqslant k + m \leqslant 2 \dim Y + 1$ if $\mathbb{F} = \mathbb{C}$) and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\sum_{i=1}^{k} \lambda_i \langle a_i^*, g \rangle + \sum_{j=1}^{m} c_{i_j}(g) \overline{\sigma}_{i_j} = 0 \quad \text{for each} \quad g \in Y. \tag{2.21}$$

In addition, if F is totally bounded, M_{F-g_0} can be replaced with E_{F-g_0}.

Proof. We will prove the theorem only for the case when F is bounded since the case when F is totally bounded is similar.

For each $i \in I_1$, define,

$$C_i = \{g \in Y : \; c_i(g) + c_i(g_0) \in J_i\}.$$

It is clear that $g \in G$ if and only if $g - g_0 \in C \cap Y$ where $C := \bigcap_{i \in I_1} C_i$. Note that

$$\sup_{x \in F} \|x - g\| = \sup_{x \in F} \|x - g_0 - (g - g_0)\| = \sup_{x \in F - g_0} \|x - (g - g_0)\|.$$

We get that $g_0 \in P_G(F)$ if and only if $0 \in P_{C \cap Y}(F - g_0)$. We now consider the problem in the finite-dimensional space Y. Thus, by Lemma 2.1, $g_0 \in P_G(F)$ if and only if 0 is an optimal solution of the minimization problem on Y given by

$$\min f_{F-g_0}(g) \tag{2.22}$$

subject to $g \in C$. Since there exists an element $\hat{g} \in G$ satisfying $c_i(\hat{g}) \in \text{int } J_i$ for each $i \in I_1$, $\text{int} \bigcap_{i \in I_1} C_i \neq \emptyset$; hence, by Deutsch et al. [6, Proposition 3.1], the collection of convex sets $\{C_i : \; i \in I_1\}$ has the strong CHIP. From [4], the following assertions are equivalent:

(i) 0 is an optimal solution of the minimization problem (2.22) subject to $g \in C$;

(ii) there exist $y^* \in \partial f_{F-g_0}(0)$, $x_i^* \in N_{C_i}(0)$ $(i \in I_1)$ such that

$$y^* + \sum_{i \in I_1} x_i^* = 0 \quad \text{on } Y. \tag{2.23}$$

Note that

$$N_{C_i}(0) = \{x^* \in Y^* : \overline{\alpha}_i \in N_{J_i}(c_i(g_0)),\ \alpha_j = 0\ \forall j \in I \setminus I_0,\ j \neq i\}, \tag{2.24}$$

where $\alpha_j := \langle x^*, y_j \rangle$ for $j \in I$. Clearly $i \notin I(g_0)$ if and only if $c_i(g_0) \in \operatorname{int} J_i$. Hence, $N_{J_i}(c_i(g_0)) = 0$ and $N_{C_i}(0) = 0$ in the case when $i \notin I(g_0)$. By (2.24), in the case when $i \in I(g_0)$, $x_i^* \in N_{C_i}(0)$ if and only if there exists $\alpha_i \in N_{J_i}(c_i(g_0))$ such that

$$\langle x_i^*, g \rangle = c_i(g)\overline{\alpha}_i \quad \text{for each } g \in Y. \tag{2.25}$$

Moreover, by Lemma 2.2, y^* can be expressed as

$$-y^* = \sum_{i=1}^{k} \mu_i a_i^* \quad \text{on } Y \tag{2.26}$$

for some $a_i^* \in M_{F-g_0}$, $(i = 1, 2, \ldots, k)$ and $\mu_i > 0$ with $\sum_{i=1}^{k} \mu_i = 1$. Hence, by (2.25) and (2.26), (ii) holds if and only if there exist $a_i^* \in M_{F-g_0}$, $\lambda_i > 0$ $(i = 1, 2, \ldots, k,\ k > 0)$ and $\alpha_{i_j} \in \sigma_i(g_0)$ $(i_j \in I(g_0),\ j = 1, 2, \ldots, m)$ such that

$$\sum_{i=1}^{k} \lambda_i \langle a_i^*, g \rangle + \sum_{j=1}^{m} c_{i_j}(g)\overline{\alpha_{i_j}} = 0 \quad \text{for each } g \in Y. \tag{2.27}$$

Moreover, when (2.27) hold, we can have the additional property:

$$1 + m \leqslant k + m \leqslant \begin{cases} \dim Y + 1 & \text{if } \mathbb{F} = \mathbb{R}, \\ 2\dim Y + 1 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases} \tag{2.28}$$

In fact, assume that $\dim Y = l$ and, without loss of generality, let $\{y_1, \ldots, y_l\}$ be a basis of Y. Let $\mathcal{V}$ and $\mathcal{U}$ denote the convex hull of the sets $\{(\langle a_i^*, y_1 \rangle, \langle a_i^*, y_2 \rangle, \ldots, \langle a_i^*, y_l \rangle) : i = 1, 2, \ldots, k\}$ and $\{(c_{i_j}(y_1), c_{i_j}(y_2), \ldots, c_{i_j}(y_l))\overline{\alpha_{i_j}} : j = 1, 2, \ldots, m\}$, respectively. Since there exists an element $\hat{g} \in G$ satisfying $c_i(\hat{g}) \in \operatorname{int} J_i$ for each $i \in I_1$, we have that $0 \notin \operatorname{co} \mathcal{U}$. Hence, (2.27) holds if and only if $0 \in \operatorname{co}(\mathcal{V} \cup \mathcal{U})$. Thus, by the Caratheodory Theorem (cf. [2]), we can select subsets of $\{a_1^* : i = 1, 2, \ldots, k\}$ and $\{\alpha_{i_j} : j = 1, 2, \ldots, m\}$, denoted by themselves, such that (2.27) and (2.28) are satisfied. As we already noted, $g_0 \in P_G(F)$ if and only if (i) holds. Therefore, the proof is complete. □

Now let us introduce the concept of *RS*-sets in Banach spaces over the field $\mathbb{F}$.

Definition 1. An n-dimensional subspace Z of a Banach space X over the field $\mathbb{F}$ is called an interpolating subspace (*resp*. a strictly interpolating subspace) if no nontrivial linear combination of n linearly independent points from the set $\operatorname{ext} \mathbf{B}^*$ (*resp*. $\overline{\operatorname{ext} \mathbf{B}^*}$) annihilates G.

Remark 1. Note that An n-dimensional subspace Z of a Banach space X over the field $\mathbb{F}$ is an interpolating subspace (*resp*. a strictly interpolating subspace) if and only if for any n linearly independent points $x_i^*, \dots, x_n^*$ from the set ext $\mathbf{B}^*$ (*resp*. $\overline{\text{ext}\,\mathbf{B}^*}$) and any n scalars $c_1, \dots, c_n \in \mathbb{F}$ there exists uniquely an element $g \in Z$ such that

$$\langle x_i^*, g\rangle = c_i \quad \text{for each } i = 1, 2 = \cdots, n.$$

Definition 2. Let X be a Banach space over the field $\mathbb{F}$ and let $\{y_1, y_2, \dots, y_n\}$ be n linearly independent elements of X. We call the set G defined by (2.19) an *RS*-set (resp. a strict *RS*-set) if every subset of $\{y_1, y_2, \dots, y_n\}$ consisting of all y_i with J_i of type (I) and some y_i with J_i of type (II) spans an interpolating subspace (resp. a strictly interpolating subspace).

Remark 2. In the case of $\mathbb{F} = \mathbb{R}$, the definition of an *RS*-set was introduced by Amir [1]. However, in the case of $\mathbb{F} = \mathbb{C}$, it seems the first time that the notion of an *RS*-set is introduced.

Finally we still need the following lemma, which plays an important role in the coming two sections.

Lemma 3. *Suppose G is a strict RS-set* (resp. *an RS-set*) *in X over field* $\mathbb{F}$. *Let* $F \subset X$ *be a bounded subset* (resp. *a totally bounded subset*) *satisfying* $r_G(F) > r_X(F)$ *and* $g_0 \in P_G(F)$. *Let* $A(F - g_0) = \{a_1^*, \dots, a_k^*\} \subseteq M_{F-g_0}$ (resp. E_{F-g_0}) *and* $B(g_0) = \{i_1, \dots, i_m\} \subseteq I(g_0)$ *with positive numbers* $\lambda_1, \dots, \lambda_l$ *satisfy* (2.21). *Then there are at least* $\dim Y - m$ *linearly independent elements in* $A(F - g_0)$.

Proof. As before, we prove the lemma only for the case when F is bounded. Set

$$Q = \{g \in Y : \ c_{i_j}(g) = 0, \ j = 1, \dots, m\}. \tag{2.29}$$

Then Q is an strictly interpolating subspace of dimension $N = \dim Y - m$. With no loss of generality, we may assume that $a_1^*, \dots, a_{k'}^*$ are linearly independent and (2.21) can be rewritten into

$$\sum_{i=1}^{k'} \lambda_i' \langle a_i^*, g\rangle + \sum_{j=1}^{m} \sigma_{i_j}(g_0) c_{i_j}(g) = 0 \quad \text{for each} \quad g \in Y. \tag{2.30}$$

To complete the proof, it suffices to show that $k' \geqslant N$. Suppose on the contrary that $k' < N$. Since Q is a strictly interpolating subspace of dimension $N = \dim Y - m$, by Remark 2.1, there exists an element $q_0 \in Q \setminus \{0\}$ such that

$$\langle a_i^*, q_0\rangle = \overline{\lambda_i'}, \ i = 1, \dots, k',$$

which with (2.30) implies $\sum_{i=1}^{k'} |\lambda'{}_i|^2 = 0$ and so $\lambda'_i = 0$ for each $i = 1, 2, \ldots, k'$. Consequently,

$$\sum_{i=1}^{k} \lambda_i \langle a_i^*, g\rangle = \sum_{i=1}^{k'} \lambda'_i \langle a_i^*, g\rangle = 0 \quad \text{for all } g \in X.$$

This implies that $g_0 \in P_X(F)$, which contradicts that $r_G(F) > r_X(F)$. The proof is complete. □

3. Strong uniqueness in real Banach spaces

In this section we always assume that $\mathbb{F} = \mathbb{R}$, i.e., X is a real Banach space X. We begin with a general theorem on the strong uniqueness of the restricted Chebyshev center with respect to a finite-dimensional polyhedron.

Definition 4. A closed convex subset G of X is called a polyhedron if it is the intersection of a finite number of closed half-spaces, that is,

$$G = \bigcap_{i=1}^{k} \{x \in X : \quad \langle x_i^*, x\rangle \leqslant d_i\}$$

for some $x_i^* \in X^* \setminus \{0\}$ and real scalars d_i. A closed convex subset G of X is called a polyhedron of finite dimension if it is the intersection of a polyhedron and a finite-dimensional subspace of X.

Theorem 1. *Let G be a polyhedron of finite dimension of X. Let $F \subset X$ be bounded and $g_0 \in P_G(F)$. Suppose that the strict Kolmogorov condition*

$$\max_{a^* \in M_{F-g_0}} \langle a^*, g_0 - g\rangle > 0 \quad \text{for each } g \in G \setminus \{g_0\} \tag{3.1}$$

holds. Then g_0 is strongly unique, that is, there exists a constant $r = r(F) > 0$ such that

$$\sup_{x \in F} \|x - g\| \geqslant \sup_{x \in F} \|x - g_0\| + r\|g - g_0\| \quad \text{for each } g \in G. \tag{3.2}$$

Proof. Assume that

$$G = \bigcap_{i=1}^{k} \{x \in X : \quad \langle x_i^*, x\rangle \leqslant d_i\}$$

for some $x_i^* \in X^* \setminus \{0\}$ and real scalars d_i. For convenience, we write $I = \{1, 2, \ldots, k\}$,

$$I_0 = \{i \in I : \langle x_i^*, g\rangle = d_i \text{ for all } g \in G\} \tag{3.3}$$

and

$$H_i = \{g \in G : \langle x_i^*, g\rangle = d_i\} \quad \text{for each } i \in I. \tag{3.4}$$

For $g \in G$, set

$$J(g) = \{i \in I : \ g \in H_i\} \tag{3.5}$$

and, if $J(g_0) \neq I$,

$$G_0 = \bigcup_{i \notin J(g_0)} H_i.$$

Note that, in the case when $J(g_0) \neq I$, G_0 is a nonempty closed subset of G and $g_0 \notin G_0$; hence $t^* = d(g_0, G_0) > 0$. For each $g \in G \setminus \{g_0\}$ and each $\lambda > 0$, define

$$T_\lambda(g) = \left(1 - \frac{\lambda}{\|g_0 - g\|}\right) g_0 + \frac{\lambda}{\|g_0 - g\|} g.$$

Set

$$\lambda^* = \begin{cases} t^* & \text{if } J(g_0) \neq I, \\ 1 & \text{if } J(g_0) = I. \end{cases}$$

First, we will show that

$$T_{\lambda^*}(g) \in G \quad \text{for each } g \in G. \tag{3.6}$$

Indeed, it is trivial in the case when $J(g_0) = I$. Therefore, we may assume that $J(g_0) \neq I$. Let $I^+ = \{i \in I : \ \langle x_i^*, T_{t^*}(g)\rangle > d_i\}$. Suppose on the contrary that $T_{t^*}(g) \notin G$. Since $\langle x_i^*, T_{t^*}(g)\rangle \leqslant d_i$ for each $i \in J(g_0)$, we have that

$$I^+ \neq \emptyset \quad \text{and} \quad I^+ \bigcap J(g_0) = \emptyset. \tag{3.7}$$

For each $i \in I^+$, let $0 < \lambda_i < t^*$ satisfy $\langle x_i^*, T_{\lambda_i}(g)\rangle = d_i$ and let $\lambda = \min_{i \in I^+} \lambda_i$. Then, for each $i \in I^+$,

$$\langle x_i^*, T_{\lambda_i}(g)\rangle \leqslant d_i. \tag{3.8}$$

Clearly, (3.8) holds for each $i \in I \setminus I^+$ since $0 < \lambda < t^*$. Hence $T_\lambda(g) \in G$. Moreover, by the definition of λ, we have that $J(T_\lambda(g)) \cap I^+ \neq \emptyset$. Take $i_0 \in J(T_\lambda(g)) \cap I^+$. Then $T_\lambda(g) \in H_{i_0}$. By (3.7), $i_0 \notin J(g_0)$. Hence $T_\lambda(g) \in G_0$. This implies that

$$t^* = d(g_0, G_0) \leqslant \|g_0 - T_\lambda(g)\| = \lambda < t^*,$$

which is a contradiction, This proves that $T_{t^*}(g) \in G$.

Secondly, we will show that

$$\gamma = \inf_{g \in G \setminus \{g_0\}} \max_{a^* \in M_{F-g_0}} \frac{\lambda^* \langle a^*, g_0 - g\rangle}{\|g_0 - g\|} > 0. \tag{3.9}$$

For this purpose, let

$$\gamma(g) = \max_{a^* \in M_{F-g_0}} \frac{\lambda^* \langle a^*, g_0 - g\rangle}{\|g_0 - g\|} \tag{3.10}$$

and suppose on the contrary that there exists a sequence $\{g_n\} \subset G \setminus \{g_0\}$ such that $\gamma(g_n) \to 0$ as $n \to \infty$. Due to the compactness, we may assume that $\frac{\lambda^*(g_0-g_n)}{\|g_0-g_n\|} \to \tilde{g} \neq 0$. Since $g_0 - \frac{\lambda^*(g_0-g_n)}{\|g_0-g_n\|} = T_{\lambda^*}(g_n)$, by (3.6), $g_0 - \frac{\lambda^*(g_0-g_n)}{\|g_0-g_n\|} \in G$. Consequently, $g_0 - \tilde{g} \in G \setminus \{g_0\}$. However,

$$\max_{a^* \in M_{F-g_0}} \langle a^*, g_0 - (g_0 - \tilde{g}) \rangle = \lim_{n\to\infty} \gamma(g_n) = 0, \tag{3.11}$$

which contradicts to (3.1). Hence (3.9) holds.

Finally, from (3.9) we have that

$$\max_{a^* \in M_{F-g_0}} \langle a^*, g_0 - g \rangle \geqslant \frac{\gamma\|g_0 - g\|}{\lambda^*} \quad \text{for each } g \in G. \tag{3.12}$$

Let $a_0^* \in M_{F-g_0}$ be such that

$$\langle a_0^*, g_0 - g \rangle = \max_{a^* \in M_{F-g_0}} \langle a^*, g_0 - g \rangle. \tag{3.13}$$

Thus, by Lemma 2.1, (3.12) and (3.13), we have that

$$\begin{aligned} \sup_{x\in F} \|x - g\| &\geqslant U_F^+(a_0^*) - \langle a_0^*, g \rangle \\ &= U_F^+(a_0^*) - \langle a_0^*, g_0 \rangle + \langle a_0^*, g_0 - g \rangle \\ &\geqslant \sup_{x\in F} \|x - g_0\| + \frac{\gamma}{\lambda^*}\|g_0 - g\| \end{aligned}$$

for all $g \in G$, that is, g_0 is strongly unique. The proof of the theorem is complete. □

Now we are ready to give the main theorem of this section.

Theorem 2. *Suppose that G is a real strict RS-set* (resp. *a real RS-set*) *and that $F \subset X$ is a bounded subset* (resp. *a totally bounded subset*) *satisfying $r_G(F) > r_X(F)$. Then the restricted Chebyshev center of F with respect to G is strongly unique.*

Proof. We prove the theorem just for the case when F is bounded. Let $g_0 \in P_G(F)$. By Theorem 3.1, we only need to show that the strict Komogorov condition (3.1) holds. Suppose on the contrary that there exists $g_1 \in G \setminus \{g_0\}$ such that

$$\max_{a^* \in M_{F-g_0}} \langle a^*, g_0 - g_1 \rangle \leqslant 0. \tag{3.14}$$

Then from Theorem 2.1, there exist $A(F - g_0) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subseteq M_{F-g_0}$, $B(g_0) = \{i_1, i_2, \ldots, i_m\} \subseteq I(g_0)$, $\sigma_{i_j} \in \sigma_{i_j}(g_0)$, $j = 1, \ldots, m$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$\sum_{i=1}^{k} \lambda_i \langle a_i^*, g_1 - g_0 \rangle + \sum_{j=1}^{m} c_{i_j}(g_1 - g_0)\sigma_{i_j} = 0. \tag{3.15}$$

By the definition of $\sigma_{i_j}(g_0)$, $\sigma_{i_j} \neq 0$ and $\sigma_{i_j} c_{i_j}(g_0 - g_1) \leqslant 0$, $j = 1, \ldots, m$. It follows that

$$0 \geqslant \sum_{i=1}^{k} \lambda_i \langle a_i^*, g_0 - g_1 \rangle = -\sum_{j=1}^{m} \sigma_{i_j} c_{i_j}(g_0 - g_1) \geqslant 0.$$

This implies that

$$\langle a_i^*, g_0 - g_1 \rangle = 0, \quad i = 1, \ldots, k; \tag{3.16}$$

$$c_{i_j}(g_0 - g_1) = 0, \quad j = 1, \ldots, m. \tag{3.17}$$

Hence, $g_0 - g_1 \in Q$, where Q is defined by (2.29). Since G is a strict *RS*-set, by Lemma 2.3, $A(F - g_0)$ contains at least $\dim Y - m$ linearly independent elements. Moreover, Q is a strictly interpolating subspace of dimension $\dim Y - m$. Therefore $g_0 = g_1$, which contradicts that $g_1 \in G \setminus \{g_0\}$. The proof is complete. □

Remark 1. In the case when G is a strictly interpolating subspace (*resp.* an interpolating subspace) of X. Theorem 3.2 was proved independently in [8,10].

4. Generalized strong uniqueness in complex Banach spaces

In this section we always assume that $\mathbb{F} = \mathbb{C}$, i.e., X is a complex Banach space. We begin with a counter-example which illustrates that the restricted Chebyshev center with respect to an *RS*-set is, in general, not unique in a complex Banach space.

Example 2. Let $Q = \{-1, 0, 1\}$ and $X = C(Q)$, the complex continuous function space defined on Q with the uniform norm. Define

$$g_1(t) = 1, \quad g_2 = t - \frac{1}{2} \quad \forall t \in Q,$$

$$J_1 = \{z \in C : \operatorname{Re} z \geqslant 1\}, \quad J_2 = \{z \in C : \operatorname{Re} z \leqslant 1\}$$

and

$$f(t) = \begin{cases} -\frac{1}{2} & t = -1, \\ 0 & t = 0, \\ \frac{3}{2} & t = 1. \end{cases}$$

Then, $G = \{g = c_1 + c_2\left(t - \frac{1}{2}\right) : \operatorname{Re} c_1 \geqslant 1,\ \operatorname{Re} c_2 \leqslant 1\}$. Take

$$g_1^* = 1 + \left(t - \frac{1}{2}\right), \quad g_2^* = 1 + \frac{\mathrm{i}}{8} + \left(1 + \frac{\mathrm{i}}{4}\right)\left(t - \frac{1}{2}\right).$$

Obviously, $\|f - g_1^*\| = |(f - g_1^*)(0)| = \frac{1}{2}$. Since, for any $g = c_1 + c_2\left(t - \frac{1}{2}\right) \in G$,

$$\|f - g\| \geqslant |(f - g)(0)| = \left|c_1 - \frac{1}{2}c_2\right| \geqslant \frac{1}{2},$$

we have that $g_1^* \in P_G(f)$. On the other hand, it is easy to check that

$$\|f - g_2^*\| \leqslant |(f - g_2^*)(0)| = \frac{1}{2}$$

so that $g_2^* \in P_G(f)$.

However, the following theorem shows that, under some additional convexity condition on J_i, the restricted Chebyshev center with respect to an *RS*-set is unique. Recall that a convex subset J of $\mathbb{C}$ is strictly convex if, for any two distinct elements $z_1, z_2 \in J$, $\frac{1}{2}(z_1 + z_2) \in \operatorname{int} J$.

Theorem 1. *Suppose that G is a strict RS-set* (resp. *an RS-set*) *and that* $F \subset X$ *is a bounded subset* (resp. *a totally bounded subset*) *satisfying* $r_G(F) > r_X(F)$. *If each* J_i $(i \in I_1)$ *is strictly convex, then the restricted Chebyshev center of F with respect to G is unique*.

Proof. Suppose that $P_G(F)$ has two distinct elements g_1, g_2. Write $g_0 = (g_1 + g_2)/2$. Then, using standard techniques, we have that

$$M_{F-g_0} \subseteq M_{F-g_1} \cap M_{F-g_2} \subseteq \{a^* \in \overline{\operatorname{ext} \mathbf{B}^*} : \ \langle a^*, g_1 - g_2 \rangle = 0\} \tag{4.1}$$

and, by the strict convexity of each J_i $(i \in I_1)$,

$$I(g_0) \subseteq I(g_1) \cap I(g_2) \subseteq \{i : \ c_i(g_1 - g_2) = 0\}. \tag{4.2}$$

Let

$$Q_0 = \{g \in P : \ c_i(g) = 0, \ \ i \in I(g_0)\}. \tag{4.3}$$

In view of the definition of a strict *RS*-set, Q_0 is a strictly interpolating subspace of dimension $\dim Y - |I(g_0)|$, where $|I(g_0)|$ denotes the cardinality of the set $I(g_0)$. Clearly, $g_1 - g_2 \in Q_0$. It follows from Lemma 2.3 that M_{F-g_0} contains at least $\dim Y - |I(g_0)|$ linearly independent elements. Hence $g_1 - g_2 = 0$ and the proof is complete. □

Now let us consider the problem of the strong uniqueness of the restricted Chebyshev center with respect to G. We first introduce the following definition of strong uniqueness of order α, see, for example, [15,18,21].

Definition 1. Let G be a closed nonempty subset of a Banach space X and F a bounded subset of X. Let $g_0 \in P_G(F)$. Then g_0 is called strongly unique of order $\alpha > 0$ if there exists a constant $c_\alpha = c_{\alpha,F} > 0$ such that

$$\sup_{x \in F} \|x - g\|^\alpha \geqslant \sup_{x \in F} \|x - g_0\|^\alpha + c_\alpha \|g - g_0\|^\alpha \quad \text{for each } g \in G. \tag{4.4}$$

The following result, which is trivial in the case when F is totally bounded, will be useful.

Lemma 1. *Let F be a bounded* (resp. *totally bounded*) *subset of X and* $g_0 \in P_G(F)$. *Then, for any* $a^* \in M_{F-g_0}$ (resp. E_{F-g_0}) *and* $g \in G$,

$$\sup_{x\in F} \|x-g\|^2 \geqslant r_G(F)^2 + |\langle a^*, g_0-g\rangle|^2 + 2r_G(F)\mathrm{Re}\langle a^*, g_0-g\rangle. \tag{4.5}$$

Proof. Let $a^* \in M_{F-g_0}$ (*resp.* E_{F-g_0}) and $g \in G$. Then

$$U_F^+(a^*) - \mathrm{Re}\langle a^*, g_0\rangle = \sup_{x\in F} \|x-g_0\| = r_G(F).$$

By (2.5), there exist sequences $\{a_k^*\} \subseteq \overline{\mathrm{ext}\,\mathbf{B}^*}$ and $\{x_k\} \subseteq F$ such that

$$\lim_{k\to\infty} \mathrm{Re}\langle a_k^*, x_k-g_0\rangle = \sup_{x\in F} \|x-g_0\| = r_G(F). \tag{4.6}$$

$$\lim_{k\to\infty} \langle a_k^*, g_0-g\rangle = \langle a^*, g_0-g\rangle. \tag{4.7}$$

Note that

$$(\mathrm{Re}\langle a_k^*, x_k-g_0\rangle)^2 + (\mathrm{Im}\langle a_k^*, x_k-g_0\rangle)^2 = |\langle a_k^*, x_k-g_0\rangle|^2.$$

Taking the limits in above equality and making use of (4.6), we get

$$\lim_{k\to\infty} \mathrm{Im}\langle a_k^*, x_k-g_0\rangle = 0. \tag{4.8}$$

Consequently, by (4.6)–(4.8),

$$\lim_{k\to\infty} \mathrm{Re}\{\overline{\langle a_k^*, x_k-g_0\rangle}\cdot\langle a_k^*, g_0-g\rangle\} = r_G(F)\mathrm{Re}\langle a^*, g_0-g\rangle. \tag{4.9}$$

Hence,

$$\begin{aligned}\sup\nolimits_{x\in F} \|x-g\|^2 &\geqslant |\langle a_k^*, x_k-g\rangle|^2 \\ &= |\langle a_k^*, x_k-g_0\rangle|^2 + |\langle a_k^*, g_0-g\rangle|^2 \\ &\quad +2\,\mathrm{Re}\{\overline{\langle a_k^*, x_k-g_0\rangle}\cdot\langle a_k^*, g_0-g\rangle\}.\end{aligned} \tag{4.10}$$

Taking the limits in above inequality and making use of (4.6), (4.7) and (4.9), we have (4.4) and complete the proof. □

Theorem 2. *Let G be a strict RS-set* (resp. *an RS-set*) and let $F \subset X$ *be a bounded subset* (resp. *a totally bounded subset*) *satisfying* $r_G(F) > r_X(F)$. *Suppose that, for each* $i \in I_1$ *and each* $z^* \in \mathrm{bd}\, J_i$, $\mathrm{bd}\, J_i$ *has a positive curvature at* z^*. *Then the restricted Chebyshev center of F with respect to G is strongly unique of order* 2.

Proof. Due to the same reason, we will prove the theorem only for the case when F is bounded. Under the assumption of the theorem, each J_i is strictly convex. By Theorem 4.1, the restricted Chebyshev center of F with respect to G is unique. Let g_0 be the unique restricted Chebyshev center. From Theorem 2.1, it follows that there exist $A(F-g_0) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subseteq M_{F-g_0}$, $B(g_0) = \{i_1, i_2, \ldots, i_m\} \subseteq I(g_0), \sigma_{i_j} \in \sigma_{i_j}(g_0)$, $j = 1, \ldots, m$ $(k \geqslant 1,\ k+m \leqslant 2\dim Y + 1)$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that (2.21) holds. Without loss of generality, we may assume that $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfy $\sum_{i=1}^k \lambda_i = r_G(F)$. Set

$$\gamma(g) = \frac{\sup_{x\in F} \|x-g\|^2 - r_G(F)^2}{\|g-g_0\|^2} \quad \text{for each } g \in G \setminus \{g_0\}. \tag{4.11}$$

It is sufficient to show that $\gamma(g)$ has positive lower bounds on $G \setminus \{g_0\}$. Suppose on the contrary that there exists a sequence $\{g_n\} \subset G\setminus\{g_0\}$ such that $\gamma(g_n) \to 0$. Then $\sup_{x\in F} \|x-g_n\| \to \sup_{x\in F} \|x-g_0\|$. With no loss of generality, we may assume that $g_n \to g_0$ due to the uniqueness of the restricted Chebyshev center. For $j = 1, 2, \ldots, m$, let $\kappa_{i_j} > 0$ and u_{i_j} denote the curvature and the center of curvature at $c_{i_j}(g_0)$, respectively. Define

$$\hat{c}_{i_j} = 2u_{i_j} - c_{i_j}(g_0), \quad r_{i_j} = 2|u_{i_j} - c_{i_j}(g_0)| = 2/\kappa_{i_j} \quad \text{for each } j = 1, 2, \ldots, m. \tag{4.12}$$

Then there exists a neighborhood U_{i_j} of $c_{i_j}(g_0)$ such that

$$|z - \hat{c}_{i_j}| \leqslant r_{i_j} \quad \text{for each } z \in J_{i_j} \cap U_{i_j} \text{ and each } j = 1, 2, \ldots, m. \tag{4.13}$$

Clearly, for each $i_j \in B(g_0)$ and $\sigma_{i_j} \in \sigma_{i_j}(g_0)$, $\sigma_{i_j} = d_{i_j}(\hat{c}_{i_j} - c_{i_j}(g_0))$ for some $d_{i_j} > 0$. Thus, by (2.21),

$$\sum_{i=1}^k \lambda_i \langle a_i^*, g_0 - g_n\rangle + \sum_{j=1}^m d_{i_j}\, c_{i_j}(g_0 - g_n)\overline{(\hat{c}_{i_j} - c_{i_j}(g_0))} = 0 \quad \text{for each } n = 1, 2, \ldots. \tag{4.14}$$

In addition, by (4.13), we also have that

$$|c_{i_j}(g_n) - \hat{c}_{i_j}| \leqslant r_{i_j} \quad \text{for each } i_j \in B(g_0) \tag{4.15}$$

holds for all n large enough since $c_{i_j}(g_n) \to c_{i_j}(g_0)$ as $n \to \infty$. Now define

$$\|g\|_2 = \left(\sum_{i=1}^k \lambda_i |\langle a_i^*, g\rangle|^2 + \sum_{j=1}^m d_{i_j} |c_{i_j}(g)|^2\right)^{1/2} \quad \text{for each } g \in Y. \tag{4.16}$$

By Lemma 2.3, it is easy to verify that $\|\cdot\|_2$ is a norm on Y so that it is equivalent to the original norm. Consequently, there exists a constant $\gamma > 0$ such that

$$\|g\|_2 \geqslant \gamma \|g\| \quad \text{for each } g \in Y. \tag{4.17}$$

Since $\sum_{i=1}^k \lambda_i = r_G(F)$, by Lemma 4.1, for each n,

$$\sup_{x\in F} \|x - g_n\|^2 \geqslant \frac{1}{r_G(F)} \left\{ \sum_{i=1}^k \lambda_i r_G(F)^2 + \sum_{i=1}^k \lambda_i |\langle a_i^*, g_0 - g_n\rangle|^2 \right\} + 2\sum_{i=1}^k \lambda_i \mathrm{Re}\langle a_i^*, g_0 - g_n\rangle. \tag{4.18}$$

Because $|\hat{c}_{i_j} - c_{i_j}(g_0)| = r_{i_j}$, by (4.14) and (4.15), we get that, for n large enough,

$$\begin{aligned} 2\sum_{l=1}^{k} \lambda_i \mathrm{Re}\langle a_i^*, g_0 - g_n\rangle \\ &\geqslant 2\sum_{i=1}^{k} \lambda_i \mathrm{Re}\langle a_i^*, g_0 - g_n\rangle + \sum_{j=1}^{m} d_{i_j}|\hat{c}_{i_j} - c_{i_j}(g_n)|^2 - \sum_{j=1}^{m} d_{i_j} r_{i_j}^2 \\ &= \sum_{j=1}^{m} d_{i_j}|c_{i_j}(g_0) - c_{i_j}(g_n)|^2. \end{aligned} \quad (4.19)$$

Hence, by (4.18), (4.19) and (4.17),

$$\begin{aligned} \sup_{x\in F} \|x - g_n\|^2 &\geqslant r_G(F)^2 + \frac{1}{r_G(F)} \sum_{l=1}^{k} \lambda_l |\langle a_l^*, g_0 - g_n\rangle|^2 \\ &\quad + \sum_{j=1}^{m} d_{i_j}|c_{i_j}(g_0) - c_{i_j}(g_n)|^2 \\ &\geqslant r_G(F)^2 + \min\left\{\frac{1}{r_G(F)}, 1\right\} \|g_0 - g_n\|_2^2 \\ &\geqslant r_G(F)^2 + \min\left\{\frac{1}{r_G(F)}, 1\right\} \gamma^2 \|g_0 - g_n\|^2. \end{aligned} \quad (4.20)$$

This means that $\gamma(g_n) \geqslant \min\left\{\frac{1}{r_G(F)}, 1\right\}\gamma^2$ for n large enough, which contradicts that $\gamma(g_n) \to 0$. The proof is complete. $\square$

In order to give some more general strong uniqueness theorems, recall that for each closed convex subset J_i of $\mathbb{C}$ with nonempty interior there exists a convex function f_i on $\mathbb{C}$ such that

$$\operatorname{int} J_i = \{z \in \mathbb{C} : f_i(z) < 0\} \quad \text{and} \quad \operatorname{bd} J_i = \{z \in \mathbb{C} : f_i(z) = 0\}. \quad (4.21)$$

Moreover, we require the notion of uniformly convex function and some useful properties, see, for example, [22].

Definition 2. A function $f : \mathbb{C} \to \mathbb{R}$ is uniformly convex at $z^* \in \mathbb{C}$ if there exists $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(t) > 0$ for $t > 0$ such that

$$f(\lambda z^* + (1-\lambda)z) \leqslant \lambda f(z^*) + (1-\lambda)f(z) - \lambda(1-\lambda)\delta(|z^* - z|)$$
$$\text{for each } z \in \mathbb{C} \text{ and each } 0 < \lambda < 1. \quad (4.22)$$

Define the modulus of convexity of f at z^*

$$\mu_{z^*}(t) = \inf\left\{\frac{\lambda f(z^*) + (1-\lambda)f(z) - f(\lambda z^* + (1-\lambda)z)}{\lambda(1-\lambda)} : \ z \in \mathbb{C}, \ |z^* - z| = t, \ 0 < \lambda < 1\right\}. \quad (4.23)$$

Clearly, F is uniformly convex at z^* if and only if $\mu_{z^*}(t) > 0$ for $t > 0$.

Definition 3. A function $f : \mathbb{C} \to \mathbb{R}$ has the modulus of convexity of order $p > 0$ at $z^* \in \mathbb{C}$ if there exists $\mu_p > 0$ such that $\mu_{z^*}(t) > \mu_p t^p$ for $t > 0$.

Proposition 1. *A function $f : \mathbb{C} \to \mathbb{R}$ has the modulus of convexity of order $p > 0$ at $z^* \in \mathbb{C}$ if and only if there exists $\mu_p > 0$ such that*

$$f(z) \geqslant f(z^*) + \operatorname{Re}(z - z^*)\overline{u} + \mu_p |z - z^*|^p \quad \text{for each } z \in \mathbb{C} \text{ and each } u \in \partial f(z^*). \tag{4.24}$$

Theorem 3. *Let G be a strict RS-set* (resp. *an RS-set*) *and let $F \subset X$ be a bounded subset* (resp. *a totally bounded subset*) *satisfying $r_G(F) > r_X(F)$. Suppose that, for any $i \in I_1$, there exists a convex function f_i satisfying* (4.21) *such that $f_i(\cdot)$ has the modulus of convexity of order $p > 0$ at each $z^* \in \operatorname{bd} J_i$. Then the restricted Chebyshev center of F with respect to G is strongly unique of order $\alpha = \max\{p, 2\}$.*

Proof. The proof is similar to that of Theorem 4.2. We assume that $g_0 \in P_G(F)$ is the unique restricted Chebyshez center since under the conditions of the theorem each J_i is clearly strictly convex. Let $A(F - g_0) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subseteq M_{F-g_0}$, $B(g_0) = \{i_1, i_2, \ldots, i_m\} \subseteq I(g_0)$, $\sigma_{i_j} \in \sigma_{i_j}(g_0)$, $j = 1, \ldots, m$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ be such that (2.21) holds and $\sum_{i=1}^k \lambda_i = r_G(F)$. As in the proof of Theorem 4.2, set

$$\gamma_\alpha(g) = \frac{\sup_{x\in F} \|x - g\|^\alpha - r_G(F)^\alpha}{\|g - g_0\|^\alpha} \quad \text{for each } g \in G \setminus \{g_0\} \tag{4.25}$$

and suppose that $\{g_n\} \subset G \setminus \{g_0\}$ such that $\gamma_\alpha(g_n) \to 0$ and $g_n \to g_0$ as $n \to \infty$. Since, for each $i_j \in B(g_0)$, $c_{i_j}(g_0)$ is not a minimizer of f_{i_j} (hence $0 \notin \partial f(c_{i_j}(g_0))$), by Clarke [3, Corollary 1, p. 56], $N_{J_{i_j}}(c_{i_j}(g_0))$ is equal to the cone generated by $\partial f(c_{i_j}(g_0))$. Consequently, for each $i_j \in B(g_0)$ and $\sigma_{i_j} \in \sigma_{i_j}(g_0)$, we have that $\sigma_{i_j} = -d_{i_j}\alpha_{i_j}$ for some $d_{i_j} > 0$ and $\alpha_{i_j} \in \partial f_{i_j}(c_{i_j}(g_0))$. Thus, from (2.21), it follows that

$$\sum_{i=1}^k \lambda_i \langle a_i^*, g_0 - g_n\rangle + \sum_{j=1}^m d_{i_j} c_{i_j}(g_n - g_0)\overline{\alpha}_{i_j} = 0 \quad \text{for each } n = 1, 2, \ldots. \tag{4.26}$$

Noting that $c_{i_j}(g_n) \in J_{i_j}$, $c_{i_j}(g_0) \in \operatorname{bd} J_{i_j}$ and $B(g_0)$ is finite, we get that, by (4.24), there exists $\mu_p > 0$ such that, for each $i_j \in B(g_0)$,

$$\operatorname{Re}((c_{i_j}(g_n) - c_{i_j}(g_0))\overline{\alpha_{i_j}}) + \mu_p |c_{i_j}(g_0) - c_{i_j}(g_n)|^p \leqslant 0 \quad \text{for each } n. \tag{4.27}$$

Thus, by (4.18), (4.26) and (4.27), one has

$$\sup_{x\in F} \|x - g_n\|^2 \geqslant \frac{1}{r_G(F)} \left\{ \sum_{i=1}^k \lambda_i r_G(F)^2 + \sum_{i=1}^k \lambda_i |\langle a_i^*, g_0 - g_n\rangle|^2 \right\} + 2\sum_{i=1}^k \lambda_i \operatorname{Re}\langle a_i^*, g_0 - g_n\rangle$$

$$
\begin{aligned}
&+2\sum_{j=1}^{m} d_{i_j}\operatorname{Re}((c_{i_j}(g_n)-c_{i_j}(g_0))\overline{\alpha_{i_j}})\\
&+2\mu_p\sum_{j=1}^{m} d_{i_j}|c_{i_j}(g_0)-c_{i_j}(g_n)|^p\\
&= r_G(F)^2+\frac{1}{r_G(F)}\sum_{i=1}^{k}\lambda_i|\langle a_l^*, g_n-g_0\rangle|^2\\
&+2\mu_p\sum_{i=1}^{m} d_{i_j}|c_{i_j}(g_0)-c_{i_j}(g_n)|^p. \qquad (4.28)
\end{aligned}
$$

Now for any $g \in Y$, define

$$\|g\|_\alpha = \left(\sum_{i=1}^{k}\lambda_i|\langle a_i^*, g\rangle|^\alpha+\sum_{j=1}^{m} d_{i_j}|c_{i_j}(g)|^\alpha\right)^{1/\alpha}. \qquad (4.29)$$

Then $\|\cdot\|_\alpha$ is a norm on Y equivalent to the original norm so that

$$\|g\|_\alpha \geqslant \gamma\|g\| \quad \text{for each } g \in Y \qquad (4.30)$$

for some constant $\gamma > 0$. Because $\|g_n - g_0\| \to 0$, by (4.28) and (4.30),

$$
\begin{aligned}
\sup_{x\in F}\|x-g_n\|^2 &\geqslant r_G(F)^2+\frac{1}{r_G(F)}\sum_{i=1}^{k}\lambda_i|\langle a_i^*, g_n-g_0\rangle|^\alpha\\
&+2\mu_p\sum_{i=1}^{m} d_{i_j}|c_{i_j}(g_0)-c_{i_j}(g_n)|^\alpha\\
&\geqslant r_G(F)^2+\min\left\{\frac{1}{r_G(F)}, 2\mu_p\right\}\|g_n-g_0\|_\alpha^\alpha\\
&\geqslant r_G(F)^2+\min\left\{\frac{1}{r_G(F)}, 2\mu_p\right\}\gamma^\alpha\|g_n-g_0\|^\alpha \qquad (4.31)
\end{aligned}
$$

holds for all n large enough. It follows from the Cauchy mean-valued theorem that

$$\sup_{x\in F}\|x-g_n\|^\alpha - r_G(F)^\alpha \geqslant \frac{\alpha}{2} r_G(F)^{\alpha-2}\left(\sup_{x\in F}\|x-g_n\|^2 - r_G(F)^2\right). \qquad (4.32)$$

Therefore, by (4.25), (4.31) and (4.32)

$$\gamma_\alpha(g_n) \geqslant \frac{\alpha}{2} r_G(F)^{\alpha-2}\min\left\{\frac{1}{r_G(F)}, 2\mu_p\right\}\gamma^\alpha > 0,$$

which contradicts that $\gamma_\alpha(g_n) \to 0$. We complete the proof. □

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